Generalized Inverse of Block Intuitionistic Fuzzy Matrices

Rajkumar Pradhan and Madhumangal Pal

Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore – 721 102, India.
e-mail: math.rajkumar@gmail.com, mmpalvu@gmail.com

Abstract

In this paper we define multiplication between intuitionistic fuzzy matrices (IFMs) and we derive the conditions for a block IFM to be regular. Also a method to find the generalized inverse of it with the help of the generalized inverses of the blocks of the original matrix is described. Again, it is shown that a block intuitionistic fuzzy matrix can be decomposed into an upper triangular idempotent intuitionistic fuzzy matrix and a lower triangular idempotent intuitionistic fuzzy matrix when the decomposition is symmetric.

Keywords: Intuitionistic fuzzy matrix (IFM), block intuitionistic fuzzy matrix (BIFM), generalized inverse (g-inverse).

1 Introduction

Partitioning of matrices is useful to effect addition and multiplication by handling smaller matrices. If we draw horizontal lines between rows and / or vertical lines between columns, the sub-matrices obtained are called blocks or cells of the matrix $A = [a_{ij}]_{n \times m}$. When a matrix is very large and it is not possible to store the entire matrix into the primary memory of a computer at a time, then matrix partition method is used to perform the operation on matrices. There are lot of advantages noted in partitioning an intuitionistic fuzzy matrix $A$ into blocks or cells. It exhibits some smaller structure of $A$ and thus save spaces. It also simplifies computations.

Atanassov [1] generalized the notion of Zadeh’s [20] fuzzy set to the concept of intuitionistic fuzzy set, which is composed of membership degree, non-membership degree and hesitation degree of an element $x$ in a set $A$. The formal definition of intuitionistic fuzzy set
(IFS) is given below:

Let $E$ be a fixed set. An intuitionistic fuzzy set (IFS) $A$ in $E$ is an object having the form $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in E \}$, where the functions $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$, which define the degree of membership and the degree of non-membership of the element $x \in E$ respectively, satisfy the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in E$.

The term $1 - \mu_A(x) - \nu_A(x) = \Pi_A(x)$ is called the degree of hesitation or indeterminacy.

For example, consider the universe $X = \{10, 25, 50, 75, 100\}$. The intuitionistic fuzzy set “Small” of $X$ denoted by $S$ may be defined as,

$S = \{ <10, 1, 0>, <25, 0.8, 0.1>, <50, 0.5, 0.5>, <75, 0.1, 0.88>, <100, 0.01, 0.9> \}.$

The intuitionistic fuzzy set “Big” of $X$ denoted by $B$ may be defined as,

$B = \{ <10, 0.01, 0.9>, <25, 0.1, 0.88>, <50, 0.5, 0.5>, <75, 0.8, 0.1>, <100, 1, 0> \}.$


In this paper, we obtain the conditions for a block intuitionistic fuzzy matrix to be regular and propose a method to determine the $g$-inverse of it with the help of the $g$-inverses of it blocks. Here we also shown that if a block intuitionistic fuzzy matrix is idempotent then its diagonal blocks are also idempotent. Here we proved that a block intuitionistic fuzzy matrix can be decomposed into an upper triangular idempotent intuitionistic fuzzy matrix and a lower triangular idempotent intuitionistic fuzzy matrix and the decomposition is symmetric.

2 Preliminaries

In this section, some elementary aspects that are necessary for this paper are introduced.

In a fuzzy matrix, the elements of the matrix represent the membership degree only but in an intuitionistic fuzzy matrix the membership degree and non-membership degree are both represented, as follows.

**Definition 2.1 (Intuitionistic fuzzy matrices)**

An intuitionistic fuzzy matrix (IFM) $A$ of order $m \times n$ is defined as
\[ A = \{x_{ij}, (a_{ij\mu}, a_{ij\nu})\}_{m \times n} \] where \( a_{ij\mu}, a_{ij\nu} \) are called membership and non-membership values of \( x_{ij} \) in \( A \), which maintains the condition \( 0 \leq a_{ij\mu} + a_{ij\nu} \leq 1 \). For simplicity, we write \( A = \{x_{ij}, a_{ij}\}_{m \times n} \) or simply \( [a_{ij}]_{m \times n} \) where \( a_{ij} = (a_{ij\mu}, a_{ij\nu}) \).

In arithmetic operations, only the values of \( a_{ij\mu} \) and \( a_{ij\nu} \) are needed, so from here we only consider the values of \( a_{ij} = (a_{ij\mu}, a_{ij\nu}) \). All elements of an IFM are the members of \( \langle F \rangle = \{(a, b) : 0 \leq a + b \leq 1 \} \).

Practically in many situations we have to handle large order intuitionistic fuzzy matrices. If the size of the matrix is very large it can not be stored into the primary memory of a computer due to its limited capacity. This matrix is generally stored into the secondary memory. Thus, to perform any unary or binary operations on matrices, some partitions (instead of entire matrices) are loaded into primary memory. After completion of such operations the outputs are saved in the suitable portion of the secondary memory. Combining all such result the final output can be achieved. This is the motivation to partition a matrix into blocks.

**Definition 2.2 (Intuitionistic fuzzy partition matrix)**

If an IFM is divided or partitioned into smaller IFMs called blocks or cells with consecutive rows and columns separated by dotted horizontal lines of full width between rows and vertical lines of full height between columns, then the IFM is called intuitionistic fuzzy partition matrix.

The elements of intuitionistic fuzzy partition matrix are smaller IFMs. The intuitionistic fuzzy matrix whose elements are blocks obtained by partitioning is called block intuitionistic fuzzy matrix (BIFM).

Thus \( A = \begin{bmatrix} \langle a_{11\mu}, a_{11\nu} \rangle & \langle a_{12\mu}, a_{12\nu} \rangle & \cdots & \langle a_{13\mu}, a_{13\nu} \rangle & \langle a_{14\mu}, a_{14\nu} \rangle \\ \langle a_{21\mu}, a_{21\nu} \rangle & \langle a_{22\mu}, a_{22\nu} \rangle & \cdots & \langle a_{23\mu}, a_{23\nu} \rangle & \langle a_{24\mu}, a_{24\nu} \rangle \\ \langle a_{31\mu}, a_{31\nu} \rangle & \langle a_{32\mu}, a_{32\nu} \rangle & \cdots & \langle a_{33\mu}, a_{33\nu} \rangle & \langle a_{34\mu}, a_{34\nu} \rangle \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \)

where \( P_{11} = \begin{bmatrix} \langle a_{11\mu}, a_{11\nu} \rangle & \langle a_{12\mu}, a_{12\nu} \rangle \end{bmatrix} \), \( P_{12} = \begin{bmatrix} \langle a_{13\mu}, a_{13\nu} \rangle & \langle a_{14\mu}, a_{14\nu} \rangle \end{bmatrix} \),

\( P_{21} = \begin{bmatrix} \langle a_{21\mu}, a_{21\nu} \rangle & \langle a_{22\mu}, a_{22\nu} \rangle \end{bmatrix} \) and \( P_{22} = \begin{bmatrix} \langle a_{23\mu}, a_{23\nu} \rangle & \langle a_{24\mu}, a_{24\nu} \rangle \end{bmatrix} \).
The IFM $A = \begin{bmatrix} P_{11} & P_{12} \\ \vdots & \vdots \\ P_{21} & P_{22} \end{bmatrix}$ is an example of block intuitionistic fuzzy matrix.

**Definition 2.3 (Diagonal blocks)**

The blocks along the diagonal of the block intuitionistic fuzzy matrix are called diagonal blocks. The blocks $P_{ij}$ for which $i = j$ are diagonal blocks. Thus $P_{11}$ and $P_{22}$ diagonal blocks of the block intuitionistic fuzzy matrix $A = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$.

Comparision between two intuitionistic fuzzy matrices of same order is an important role in our work, which is discuss below.

**Definition 2.4 (Dominance of IFM)**

Let $A, B \in F_{m \times n}$ such that $A = (\langle a_{ij}, a_{ij} \rangle)$ and $B = (\langle b_{ij}, b_{ij} \rangle)$, then we write $A \leq B$ if, $a_{ij} \leq b_{ij}$ and $a_{ij} \geq b_{ij}$ for all $i, j$, and we say that $A$ is dominated by $B$ or $B$ dominates $A$.

$A$ and $B$ are said to be comparable, if either $A \leq B$ or $B \leq A$.

**Definition 2.5 (Multiplication by a scalar)**

Let $A = (\langle a_{ij}, a_{ij} \rangle) \in F_{m \times n}$ and $c = (c_\mu, c_\nu) \in \langle F \rangle$ be a scalar such that $0 \leq c_\mu + c_\nu \leq 1$, then the scalar multiplication is defined as $cA = (\langle \min\{c_\mu, a_{ij}\}, \max\{c_\nu, a_{ij}\} \rangle) \in F_{m \times n}$.

**Definition 2.6** Let $A = (\langle a_{ij}, a_{ij} \rangle) \in F_{m \times p}$ and $B = (\langle b_{ij}, b_{ij} \rangle) \in F_{p \times n}$, be two IFMs. Then the matrix multiplication between $A$ and $B$ is given by

$$AB = (\langle \max_k \{\min(a_{ik}, b_{kj})\}, \min_k \{\max(a_{ik}, b_{kj})\} \rangle),$$

where $k = 1, 2, \ldots, p, i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

The (generalized) inverse of an IFM is defined below.

**Definition 2.7 (Generalized inverse)**

An intuitionistic fuzzy matrix $A \in F_{m \times n}$ is said to be regular if there exists another IFM, $X \in F_{n \times m}$ such that $AXA = A$. In this case, $X$ is called a generalized inverse (g-inverse) of $A$ and it is denoted by $A^\dagger$. This type of g-inverse is called inner inverse of $A$. The inner inverse of an IFM is not unique. The set of all inner inverses is denoted by $A[1]$.

For an IFM $A$ of order $m \times n$, an IFM $G \in F_{n \times m}$ is said to be outer inverse of $A$, if
$GAG = G$ and it is denoted by $A[2]$. $G$ is said to be $\{1,2\}$ inverse or semi-inverse of $A$, if $AGA = A$ and $GAG = G$ and it is denoted by $A[1,2]$. The IFM $G$ is said to be $\{1,3\}$ inverse or a least square g-inverse of $A$ if, $AGA = A$ and $(AG)^T = AG$ and it is denoted by $A[1,3]$. Again $G$ is said to be $\{1,4\}$ inverse or a minimum norm g-inverse of $A$ if, $AGA = A$ and $(GA)^T = GA$ and it is denoted by $A[1,4]$.

3 Regular block intuitionistic fuzzy matrix

In this section we discuss the regularity of block intuitionistic fuzzy matrices.

**Definition 3.1** Let $X$ be a block intuitionistic fuzzy matrix of the form

$$X = \begin{bmatrix} A_{psq} & B_{ps(n-q)} \\ C_{(m-p)s(q)} & D_{(m-p)(n-q)} \end{bmatrix}$$

with the diagonal blocks $A$ and $D$ are regular. With respect to this partitioning a Schur complement of $A$ in $X$ is an IFM, which is denoted by $X/A$ and is defined as $X/A = D - CA^T B$, where $A^T$ is one of the g-inverses of $A$. Similarly, $X/D = A - BD^T C$ is Schur complement of $D$ in $X$. $X/A$ is an intuitionistic fuzzy matrix imply, $D + CA^T B = D$.

**Definition 3.2 (Row space and column space)**

Let $A = (\langle a_{ij}, a_{ij} \rangle) \in F_{m \times n}$ be an IFM. Then the element $\langle a_{ij}, a_{ij} \rangle$ is the $ij$th entry of $A$. Let $A_{ih}(A_{ji})$ denote the $i$th row ($j$th column) of $A$.

The row space $R(A)$ of $A$ is the subspace of $V_n$ generated by the rows $\{A_{ih}\}$ of $A$. The column space $C(A)$ of $A$ is the subspace of $V_m$ generated by the columns $\{A_{ji}\}$ of $A$.

In this section, we present several results regarding g-inverse of IFM.

**Theorem 3.3** Let $A, B, C$ be three IFMs such that $A$ is regular, $R(C) \subseteq R(A)$ and $C(B) \subseteq C(A)$, then $CA^T B$ is invariant for all choices of g-inverse of $A$.

**Proof:** Since $A$ is regular with $R(C) \subseteq R(A)$ and $C(B) \subseteq C(A)$ so, $C = CA^T A$ and $B = AA^T B$, where $A^T$ is one of the g-inverse of $A$.
Now, \( CA^\dagger B = (CA^\dagger A)A^\dagger (AA^\dagger A)B = (CA^\dagger A)(A^\dagger A)B = CA^\dagger B \), where \( A^\dagger \) and \( A_\dagger \) be two different g-inverses of \( A \).

Thus, \( CA^\dagger B \) is invariant for all choice of g-inverse of \( A \).

**Example 3.4** Let \( A = \begin{bmatrix} 0.6,0.2 & 0.5,0.4 \\ 0.7,0.3 & 0.5,0.4 \end{bmatrix} \in F_{2 \times 2} \). Any g-inverse of \( A \) can be written as \( A^{-} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) where \( a \geq 0.8, b \leq 0.2, \quad a + b \leq 1 \)

\[ \begin{bmatrix} \langle a,b \rangle & \langle e,d \rangle \\ \langle e,f \rangle & \langle g,h \rangle \end{bmatrix} \] where \( c \leq 0.6, d \geq 0.4, \quad c + d \leq 1 \)

\( g \geq 0.7, h \leq 0.3, \quad g + h \leq 1 \).

If we take \( B = \begin{bmatrix} 0.7,0.3 & 0.6,0.4 \\ 0.6,0.4 & 0.5,0.5 \end{bmatrix} \) then \( AA^{-} B = B \) holds for any \( A^{-} \). So, \( C(B) \subseteq C(A) \).

Again for \( C = \begin{bmatrix} 0.6,0.3 & 0.7,0.3 \\ 0.8,0.2 & 0.6,0.4 \end{bmatrix} \), \( CA^{-} A = C \) for any \( A^{-} \). So, \( R(C) \subseteq R(A) \).

Now, \( CA^{-} B = \begin{bmatrix} 0.6,0.3 & 0.7,0.3 \\ 0.8,0.2 & 0.6,0.4 \end{bmatrix} \begin{bmatrix} 0.8,0.1 & 0.5,0.4 \\ 0.4,0.5 & 0.7,0.2 \end{bmatrix} \begin{bmatrix} 0.7,0.3 & 0.6,0.4 \\ 0.6,0.4 & 0.5,0.5 \end{bmatrix} \)

\[ = \begin{bmatrix} 0.6,0.3 & 0.6,0.4 \\ 0.7,0.3 & 0.6,0.4 \end{bmatrix} \] and

\( \begin{bmatrix} 0.6,0.3 & 0.7,0.3 \\ 0.8,0.2 & 0.6,0.4 \end{bmatrix} \begin{bmatrix} 0.9,0.1 & 0.6,0.4 \\ 0.5,0.5 & 0.8,0.2 \end{bmatrix} \begin{bmatrix} 0.7,0.3 & 0.6,0.4 \\ 0.6,0.4 & 0.5,0.5 \end{bmatrix} \)

\[ = \begin{bmatrix} 0.6,0.3 & 0.6,0.4 \\ 0.7,0.3 & 0.6,0.4 \end{bmatrix} \] for two different \( A^{-} \) of \( A \).

This is true for any generalized inverse of \( A \). Hence \( CA^{-} B \) is invariant for any g-inverse of \( A \).

**Definition 3.5** A square intuitionistic fuzzy matrix \( A \) is said to be idempotent if it satisfy the matrix equation \( A^2 = A \).

**Theorem 3.6** Let \( X \) be a BIFM of the form \( X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) with \( R(C) \subseteq R(A) \), \( C(B) \subseteq C(A) \), \( R(B) \subseteq R(D) \), \( C(C) \subseteq C(D) \) such that \( A, B, C \) are idempotent, \( A \geq B \) and \( D \geq CA^{-} B \). Then \( X \)
is regular and has a g-inverse \( X^- = \begin{bmatrix} A^- + A^- BD^- CA^- & A^- BD^- \\ D^- CA^- & D^- \end{bmatrix} \) for some \( A^- \) of \( A \) and \( D^- \) of \( D \).

**Proof:** Since \( A \) is idempotent with \( R(C) \subseteq R(A) \) and \( C(B) \subseteq C(A) \), then \( C = CA^- A \) and \( B = AA^- B \). Since \( D \geq CA^- B \) it follows that \( D = D + CA^- B \).

Hence, \( X \) can be expressed as \( X = PQR \), where \( P = \begin{bmatrix} I & O \\ CA^- & I \end{bmatrix} \), \( Q = \begin{bmatrix} A & O \\ O & D \end{bmatrix} \) and \( R = \begin{bmatrix} I & A & B \\ O & I \end{bmatrix} \).

It can be shown that \( P \) and \( R \) both are idempotent.

Now,
\[
RQ^-P = \begin{bmatrix} I & A & B \\ O & I \end{bmatrix} \begin{bmatrix} A^- & O \\ O & D^- \end{bmatrix} \begin{bmatrix} I & O \\ O & D^- \end{bmatrix} = \begin{bmatrix} A^- + A^- BD^- CA^- & A^- BD^- \\ D^- CA^- & D^- \end{bmatrix}
\]

= \( X^- \) (say).

Now, \( XX^-X = (PQR)(RQ^-P)(PQR) = (PQR)Q^- (PQR) = \begin{bmatrix} A + BD^-C & B + BD^-D \\ C + DD^-C & D + CA^-B \end{bmatrix} \). Since, \( M/D \) is a Schur complement in \( X \) and it is an IFM, so, \( A + BD^-C = A \).

As, \( R(B) \subseteq R(D) \) and \( C(C) \subseteq C(D) \) implies \( B = BD^-D \) and \( C = DD^-C \).

Hence \( XX^-X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = X \). Thus \( X \) is regular and the IFM \( X^- \) is a g-inverse of \( X \).

**Example 3.7** Let us consider the BIFM

\[
X = \begin{bmatrix} \langle 0.8,0.2 \rangle & \langle 0.6,0.4 \rangle & : & \langle 0.7,0.2 \rangle & \langle 0.6,0.4 \rangle \\ \langle 0.6,0.4 \rangle & \langle 0.7,0.3 \rangle & : & \langle 0.6,0.4 \rangle & \langle 0.6,0.4 \rangle \\ \ldots & \ldots & : & \ldots & \ldots \\ \langle 0.7,0.3 \rangle & \langle 0.6,0.4 \rangle & : & \langle 0.8,0.2 \rangle & \langle 0.6,0.4 \rangle \\ \langle 0.7,0.2 \rangle & \langle 0.6,0.3 \rangle & : & \langle 0.7,0.2 \rangle & \langle 0.7,0.2 \rangle \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.
\]

One of the g-inverse of \( A \) is \( A^- = \begin{bmatrix} \langle 0.8,0.2 \rangle & \langle 0.5,0.5 \rangle \\ \langle 0.5,0.5 \rangle & \langle 0.7,0.2 \rangle \end{bmatrix} \) and that one for \( D \) is \( D^- = \begin{bmatrix} \langle 0.8,0.1 \rangle & \langle 0.5,0.4 \rangle \\ \langle 0.6,0.3 \rangle & \langle 0.7,0.2 \rangle \end{bmatrix} \).
Now, \( CA^{-1} B = \begin{bmatrix} \langle 0.7,0.3 \rangle & \langle 0.6,0.4 \rangle \\ \langle 0.7,0.2 \rangle & \langle 0.6,0.4 \rangle \end{bmatrix} \) and \( D \geq CA^{-1} B \).

Again, \( BD^{-1} D = \begin{bmatrix} \langle 0.7,0.2 \rangle & \langle 0.6,0.4 \rangle \\ \langle 0.6,0.3 \rangle & \langle 0.6,0.4 \rangle \end{bmatrix} = B \) implies, \( R(B) \subseteq R(D) \)

and \( DD^{-1} C = \begin{bmatrix} \langle 0.7,0.3 \rangle & \langle 0.6,0.4 \rangle \\ \langle 0.7,0.2 \rangle & \langle 0.6,0.3 \rangle \end{bmatrix} = C \) implies, \( C(B) \subseteq C(D) \).

Now, \( A^{-1} BD^{-1} = \begin{bmatrix} \langle 0.7,0.2 \rangle & \langle 0.6,0.4 \rangle \\ \langle 0.6,0.4 \rangle & \langle 0.6,0.4 \rangle \end{bmatrix} \) and \( D^{-1} CA^{-1} = \begin{bmatrix} \langle 0.7,0.3 \rangle & \langle 0.6,0.4 \rangle \\ \langle 0.7,0.2 \rangle & \langle 0.6,0.3 \rangle \end{bmatrix} \).

Then \( A^{-1} BD^{-1} CA^{-1} = \begin{bmatrix} \langle 0.7,0.3 \rangle & \langle 0.6,0.4 \rangle \\ \langle 0.6,0.4 \rangle & \langle 0.6,0.4 \rangle \end{bmatrix} \) and

\[
A^{-1} + A^{-1} BD^{-1} CA^{-1} = \begin{bmatrix} \langle 0.8,0.2 \rangle & \langle 0.6,0.4 \rangle \\ \langle 0.6,0.4 \rangle & \langle 0.7,0.2 \rangle \end{bmatrix}.
\]

So, \( X^{-1} = \begin{bmatrix} \langle 0.8,0.2 \rangle & \langle 0.6,0.4 \rangle & \langle 0.7,0.2 \rangle & \langle 0.6,0.4 \rangle \\ \langle 0.6,0.4 \rangle & \langle 0.7,0.2 \rangle & \langle 0.6,0.4 \rangle & \langle 0.6,0.4 \rangle \\ \langle 0.7,0.3 \rangle & \langle 0.6,0.4 \rangle & \langle 0.8,0.1 \rangle & \langle 0.5,0.4 \rangle \\ \langle 0.7,0.2 \rangle & \langle 0.6,0.3 \rangle & \langle 0.6,0.3 \rangle & \langle 0.7,0.2 \rangle \end{bmatrix} \) and \( XX^{-1} X = X \).

**Corollary 3.8** For the BIFM \( X \) of the form as stated in Theorem 3.6, if \( A \) is symmetric and idempotent, then the semi-inverse that is \( A[1,2] \) type g-inverse of \( A \) exists. Then under the condition \( A \geq BD^{-1} C \), we get \( A^{-1} = A^{-1} AA^{-1} \geq A^{-1} BD^{-1} CA^{-1} \). This follows that

\[
A^{-1} + A^{-1} BD^{-1} CA^{-1} = A^{-1}.
\]

Then the g-inverse of \( X \) reduces to \( X^{-1} = \begin{bmatrix} A^{-1} & A^{-1} BD^{-1} \\ D^{-1} CA^{-1} & D^{-1} \end{bmatrix} \).

**Example 3.9** Let us consider the BIFM

\[
X = \begin{bmatrix} \langle 0.8,0.2 \rangle & \langle 0.6,0.4 \rangle & \langle 0.8,0.2 \rangle \\ \langle 0.6,0.4 \rangle & \langle 0.7,0.3 \rangle & \langle 0.6,0.4 \rangle \\ \ldots & \ldots & \ldots \\ \langle 0.6,0.4 \rangle & \langle 0.7,0.3 \rangle & \langle 0.6,0.4 \rangle \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.
\]

Here as \( A \) is symmetric and idempotent so \( A \) itself an \( A[1,2] \) inverse.

Now, \( CA^{-1} B = [\langle 0.6,0.4 \rangle] \) and hence, \( D \geq CA^{-1} B \).
For $D^* = [(0.6,0.4)]$, $BD^* C = \begin{bmatrix} (0.6,0.4) & (0.6,0.4) \\ (0.6,0.4) & (0.6,0.4) \end{bmatrix}$ and then, $A \geq BD^* C$.

Again, $A^{-1}BD^* = [(0.6,0.4)(0.6,0.4)]^T$ and $D^*CA^* = [(0.6,0.4)(0.6,0.4)]$.

So the g-inverse will be, $X^- = \begin{bmatrix} (0.8,0.2) & (0.6,0.4) & (0.6,0.4) \\ (0.6,0.4) & (0.7,0.3) & (0.6,0.4) \\ (0.6,0.4) & (0.6,0.4) & (0.6,0.4) \end{bmatrix}$ and for that $XX^-X = X$ holds.

**Theorem 3.10** Let $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a BIFM with $R(C) \subseteq R(A)$, $C(B) \subseteq C(A)$, $R(B) \subseteq R(D)$ and $C(C) \subseteq C(D)$. If $X$ is regular then $A$ and $D$ are regular.

**Proof:** Let $X$ be regular and $X^- = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ be a g-inverse of $X$. Hence $XX^-X = X$.

By comparing the corresponding diagonal blocks we get,

$APA + BRA + AQC + BSC = A$ \hspace{1cm} (1)

$CPB + DRB + CQD + DSD = D$. \hspace{1cm} (2)

By the given conditions on $X$, we have $C = UA$, $B = AV$, $C = DU_1$ and $B = V_D$ for some intuitionistic fuzzy matrices $U$, $V$, $U_1$, and $V_1$.

Substituting $B = AV$ and $C = UA$ in (1), we get, $A(P + VR + QU + VSU)A = A$. Hence, $A$ is regular.

Similarly, substituting $B = V_D$ and $C = DU_1$ in (2) we can prove that $D$ is regular.

**Remark 3.11** The converse of the above theorem is not true in general. This is illustrated in the following example.

**Example 3.12** Let us consider the BIFM

$X = \begin{bmatrix} (0.5,0.4) & \cdots & (0.6,0.3) & (0.7,0.3) \\ \vdots & \cdots & \cdots & \cdots \\ (0.8,0.1) & \cdots & (0.8,0.2) & (0.6,0.4) \\ (0.7,0.2) & \cdots & (0.6,0.4) & (0.7,0.2) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$.

Here $A$ and $D$ are regular. One of the g-inverse of $A$ and $D$ are respectively

$A^- = [(0.6,0.3)]$ and $D^- = \begin{bmatrix} (0.9,0.1) & (0.5,0.5) \\ (0.5,0.5) & (0.8,0.2) \end{bmatrix}$. But $X^2 \neq X$ that is, $X$ is not idempotent. So,
Theorem 3.13 For the IFMs \( A, B, C \) of order \( m \times n \) the following statements hold:

(i) If \( R(C) \subseteq R(A) \), then \( A \) is regular \( \iff [A \ C]^T \) is regular.

(ii) If \( C(B) \subseteq C(A) \), then \( A \) is regular \( \iff [A \ B] \) is regular.

**Proof:** Since \( R(C) \subseteq R(A) \) and \( A \) is regular implies \( C = CA^{-1}A \). Now, we can verify that \([A^{-1} O] \) is a g-inverse of \([A \ C]^T \). Hence, \([A \ C]^T \) is regular.

Conversely, if \( X = [A \ C]^T \) is regular, then \( R(C) \subseteq R(A) \) implies that there exists IFM \( Y \) of order \( m \) such that \( C = YA \). Hence, \( X = [I \ Y]^T A = UA \) where, \( U = [I \ Y]^T \).

For \( U^{-} = [I \ O] \), \( U^{-}U = I \) and \( U^{-} \) is a g-inverse of \( U \). Thus \( U \) is regular.

Now, \( X = UA \) is regular implies, \((UA)X^{-}(UA) = UA \). Premultiplying with \( U^{-} \) on both sides, we get \( A(X^{-}U)A = A \). Thus \( A \) is regular.

In similar way we can prove the statement in (ii).

**Example 3.14** Let \( A = [(0.8,0.2) \ (0.5,0.5)] \) and \( C = [(0.5,0.5) \ (0.5,0.5)] \) be two IFMs of order \((1 \times 2)\). Here \( R(C) \subseteq R(A) \) holds and \( A \) is regular, one of its g-inverse is \([(0.9,0.1) \ (0.5,0.4)]^T \).

Now, \([A \ C]^T = \begin{bmatrix} 0.8,0.2 & 0.5,0.5 \\ 0.5,0.5 & 0.5,0.5 \end{bmatrix} \) is an idempotent IFM. So, \([A \ C]^T \) is regular.

Similarly we can show that \([A \ B] \) is also regular for any IFM \( B \) of order \((1 \times 2)\) with \( C(B) \subseteq C(A) \).

4 LU-Decomposition of BIFM

In this section, we shall derive the conditions for a block intuitionistic fuzzy matrix to be expressed as the product of an idempotent lower block triangular intuitionistic fuzzy matrix and an idempotent upper block triangular intuitionistic fuzzy matrix.

**Lemma 4.1** Let \( X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) be an BIFM with diagonal blocks \( A \) and \( D \) are non-zero idempotents. Then the following equivalence holds:

(i) \( CA + DC = C \iff CA = DC = C \iff R(C) \subseteq R(A) \) and \( C(C) \subseteq C(D) \).

(ii) \( AB + BD = B \iff AB = BD = B \iff R(B) \subseteq R(D) \) and \( C(B) \subseteq C(A) \).

**Proof:** Since \( A \) and \( D \) are non-zero idempotents, \( A \) itself is a g-inverse of \( A \). Hence \( A \in A[1] \)
and $D \in D[1].$ As $A$ and $D$ are regular so, $CA = DC = C \Leftrightarrow R(C) \subseteq R(A)$ and $C(C) \subseteq C(D).$

Again, $CA = DC = C \Rightarrow CA + DC = C.$

Conversely, suppose $CA + DC = C,$ then $C \geq CA$ and $C \geq DC.$ Let us claim that, $C = CA,$ for $C > CA,$ then $CA > CA^2 = CA$ is not possible.

Similarly, if $D$ is idempotent, we claim that $C = DC.$ Hence $CA + DC = C \Rightarrow CA = DC = C.$ Thus the equivalence in (i) holds.

The equivalence of (ii) can be proved in the same way.

**Example 4.2** Let the blocks of the BIFM $X$ are respectively, $A = \begin{bmatrix} 0.8,0.2 & 0.6,0.4 \\ 0.6,0.4 & 0.7,0.3 \end{bmatrix}, B = \begin{bmatrix} 0.7,0.2 & 0.6,0.4 \\ 0.6,0.4 & 0.6,0.4 \end{bmatrix}, C = \begin{bmatrix} 0.7,0.3 & 0.6,0.4 \\ 0.7,0.2 & 0.6,0.3 \end{bmatrix}$ and $D = \begin{bmatrix} 0.8,0.2 & 0.6,0.4 \\ 0.7,0.2 & 0.7,0.3 \end{bmatrix}.$

Then, $CA = \begin{bmatrix} 0.7,0.3 & 0.6,0.4 \\ 0.7,0.2 & 0.6,0.3 \end{bmatrix}$ and $DC = \begin{bmatrix} 0.7,0.3 & 0.6,0.4 \\ 0.7,0.2 & 0.6,0.3 \end{bmatrix}.$

Thus $CA = DC = C$ and $CA + DC = C.$

Again, $AB = \begin{bmatrix} 0.7,0.2 & 0.6,0.4 \\ 0.6,0.4 & 0.6,0.4 \end{bmatrix}$ and $BD = \begin{bmatrix} 0.7,0.2 & 0.6,0.4 \\ 0.6,0.4 & 0.6,0.4 \end{bmatrix}.$

Thus, $AB = BD = B$ and $AB + BD = B.$

As, $CA^\dagger A = C$ for any $g$-inverse $A^\dagger$ of $A$ imply $R(C) \subseteq R(A)$ and $DD^*C = C$ for any $g$-inverse $D^*$ of $D$ imply $C(C) \subseteq C(D).$

**Lemma 4.3** Let us consider the BIFM $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with diagonal blocks $A$ and $D$ are non-zero idempotent. Then,

(i) $X/A$ is an intuitionistic fuzzy matrix $\Leftrightarrow D \geq CAB.$

(ii) $X/D$ is an intuitionistic fuzzy matrix $\Leftrightarrow A \geq BDC.$

**Proof:** (i) From the definition of Schur complement, $X/A$ is an IFM imply $D \geq CA^\dagger B.$

Again, $A$ is an idempotent IFM so $A$ itself a $g$-inverse, that is, $A^\dagger = A.$ Hence, $D \geq CAB.$

(ii) Similarly, from the definition of Schur complement, $X/D$ is an IFM imply $A \geq BD^*C.$

As, $D$ is an idempotent IFM so $D$ itself a $g$-inverse, that is, $D^* = D.$ Hence, $A \geq BDC.$
Example 4.4 If we take the same BIFM as in the above example, then
\[
CAB = \begin{bmatrix}
(0.7,0.3) & (0.6,0.4) \\
(0.7,0.2) & (0.6,0.4)
\end{bmatrix}
\text{ and } D \geq CAB \text{ holds.}
\]
Similarly, \(BDC = \begin{bmatrix}
(0.7,0.3) & (0.6,0.4) \\
(0.6,0.4) & (0.6,0.4)
\end{bmatrix}\) and \(A \geq BDC\).

Theorem 4.5 Let us consider the BIFM \(X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) with the diagonal blocks \(A\) and \(D\) are non-zero idempotent. Then \(A \geq BC\), \(D \geq CB\), \(CA = DC = C\) and \(AB = BD = B\) holds if and only if \(X\) is idempotent.

Proof: Let \(A \geq BC\), \(D \geq CB\), \(CA = DC = C\) and \(AB = BD = B\) holds.

Then, \(A + BC = [\max\{a_{ij}, bc_{ij}\}, \min\{a_{ij}, bc_{ij}\}] = [a_{ij}, a_{ij}] = A\).

Similarly, \(D \geq CB\) implies, \(D + CB = D\).

Again by Lemma 4.1, \(CA = DC = C\) implies, \(CA + DC = C\) and \(AB = BD = B\) implies, \(AB + BD = B\).

Now, \(X^2 = \begin{bmatrix}
A^2 + BC & AB + BD \\
CA + DC & CB + D^2
\end{bmatrix} = \begin{bmatrix} A + BC & AB + BD \\ CA + DC & CB + D \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = X\).

(As \(A, D\) are idempotents and using the above deduction.)

Conversely, let \(X\) is idempotent. Then equating the corresponding blocks in \(X^2 = X\), we get, \(A + BC = A\), \(CB + D = D\), \(AB + BD = B\) and \(CA + DC = C\).

Now, \(A + BC = A\) implies, \(A \geq BC\) and \(CB + D = D\) implies, \(D \geq CB\).

Again, \(AB + BD = B\) implies, \(AB = BD = B\) and \(CA + DC = C\) implies, \(CA = DC = C\).

Hence the conditions.

The conditions on \(X\) with \(A\) and \(D\) to be non-zero idempotents are essential here. This is illustrated by the following example.

Let us consider the BIFM \(X = \begin{bmatrix}
(0,1) & (1,0) \\
(0,1) & (0,1)
\end{bmatrix}\) with
\(A = \begin{bmatrix}
(0,1) & (1,0) \\
(0,1) & (0,1)
\end{bmatrix}\), \(B = [(1,0)(0,1)]^T\), \(C = [(0,1)(1,0)]\) and \(D = [(1,0)]\).
Here $X$ is idempotent with $A$ is not idempotent but $D$ is idempotent. Then $D \geq CB$, $A = BC$, $BD = B$ and $DC = C$ but $CA \neq C$ and $AB \neq B$.

Theorem 4.6 Let $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, $X_1 = \begin{bmatrix} A & O \\ C & D \end{bmatrix}$ and $X_2 = \begin{bmatrix} A & B \\ O & D \end{bmatrix}$ be three BIFMs, then the following are equivalent.

(i) $X = X_1X_2 = X_2X_1$ with both $X_1$ and $X_2$ are idempotents.

(ii) $A$ and $D$ are idempotents; $A \geq BC$, $D \geq CB$, $CA = DC = C$ and $AB = BD = B$.

Proof: $(i) \Rightarrow (ii)$

Here $X_1$ is idempotent, that is, $X_1^2 = \begin{bmatrix} A^2 & O \\ CA + DC & D^2 \end{bmatrix} = X_1$.

Equating the corresponding blocks, we get, $A^2 = A$, $D^2 = D$ and $CA + DC = C$. That is, $A$ and $D$ are idempotents with $CA + DC = C$.

Similarly, $X_2$ is idempotent imply, $A$ and $D$ are idempotents with $AB + BD = B$.

Then using Lemma 4.1, this reduces to $CA = DC = C$ and $AB = BD = B$.

Again, $X^2 = X_1X_2X_1X_2 = X_1X_2X_1X_1 = X_1X_2X_1 = X_1X_2X_2 = X_1X_2 = X$.

So $X$ is idempotent, that is, $X^2 = \begin{bmatrix} A + BC & AB + BD \\ CA + DC & CB + D \end{bmatrix} = X$.

Equating the diagonal blocks, we get, $A + BC = A$ and $CB + D = D$ which gives, $A \geq BC$ and $D \geq CB$.

Hence $X_1X_2 = X = X_2X_1$ with $X_1$, $X_2$ idempotents give, $A$ and $D$ are idempotents, $CA = DC = C$, $AB = BD = B$, $A \geq BC$ and $D \geq CB$.

$(ii) \Rightarrow (i)$

$A$ and $D$ idempotents with, $CA = DC = C$ and $AB = BD = B$ shows that $X_1$ and $X_2$ are idempotents.

Again, $A \geq BC$ and $D \geq CB$ gives, $X$ is idempotent. So, $X = X_1X_2 = X_2X_1$.

Example 4.7 Let us consider the BIFMs
Here both $X_1$ and $X_2$ are idempotents.

Now, $X_1X_2 = \begin{bmatrix} 
\langle 0.8,0.2 \rangle & \langle 0.6,0.4 \rangle & \langle 0.7,0.2 \rangle & \langle 0.6,0.4 \rangle \\
\langle 0.6,0.4 \rangle & \langle 0.7,0.3 \rangle & \langle 0.6,0.4 \rangle & \langle 0.6,0.4 \rangle \\
\langle 0.7,0.3 \rangle & \langle 0.6,0.4 \rangle & \langle 0.8,0.2 \rangle & \langle 0.6,0.4 \rangle \\
\langle 0.7,0.2 \rangle & \langle 0.6,0.3 \rangle & \langle 0.7,0.2 \rangle & \langle 0.7,0.2 \rangle 
\end{bmatrix} = X = X_2X_1.$

Also, $BC = \begin{bmatrix} 
\langle 0.7,0.3 \rangle & \langle 0.6,0.4 \rangle \\
\langle 0.6,0.4 \rangle & \langle 0.6,0.4 \rangle 
\end{bmatrix}$ so, $A \geq BC$

and $CB = \begin{bmatrix} 
\langle 0.7,0.3 \rangle & \langle 0.6,0.4 \rangle \\
\langle 0.7,0.2 \rangle & \langle 0.6,0.4 \rangle 
\end{bmatrix}$ so, $D \geq CB$.

**Corollary 4.8** Let the BIFM $X = \begin{bmatrix} A & B \\
C & D \end{bmatrix}$ with the diagonal blocks $A$ and $D$ are non-zero idempotents. Then the following are equivalent:

(i) $X$ is idempotent.

(ii) $X$ has a commuting idempotent LU-decomposition.

**Proof:** (i) $\Rightarrow$ (ii)

$X$ is idempotent implies, $A + BC = A$ and $CB + D = D$ which gives, $A \geq BC$, $CA = DC = C$, $AB = BD = B$ and $D \geq CB$. (by Lemma 4.1)
Let \( X_1 = \begin{bmatrix} A & O \\ C & D \end{bmatrix} \) and \( X_2 = \begin{bmatrix} A & B \\ O & D \end{bmatrix} \). Then, \( X_1^2 = \begin{bmatrix} A^2 & O \\ CA + DC & D^2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} = X_1 \).

Similarly, \( X_2^2 = X_2 \). So \( X_1 \) and \( X_2 \) are idempotents.

Now, \( X_1X_2 = X = X_2X_1 \).

\[(ii) \Rightarrow (i)\]
\[X^2 = (X_1X_2)^2 = (X_2X_1)^2 = X_2^2X_1^2 = X_2X_1 = X.\]

Hence \( X \) is idempotent.

5 Conclusions

Here we derived the generalized inverse of some special type of BIFMs whose blocks satisfy some conditions. Namely, we considered the blocks of the BIFMs to be idempotents and the diagonal blocks to be non-zero regular. In our future work we shall try to generalize the method of finding the g-inverse of any BIFM.

Acknowledgement: The authors are greatful to the referees and the Editor-in-Chief for the their valuable suggestions to improve the quality of the presentation of the paper.

References


